# A Note on Parabolic Subgroups of a Coxeter Group

Dongwen Qi
Department of Mathematics, Ohio State University
231 West 18th Avenue, Columbus, OH 43210, U.S.A.

February 2, 2008

**Abstract**

The aim of this note is to prove that the parabolic closure of any subset of a Coxeter group is a parabolic subgroup. To obtain that, several technical lemmas on the root system of a parabolic subgroup are established.

**MSC 2000 Subject Classifications**: Primary 20F55; Secondary 20H15

**Key Words**: Coxeter group, root system, canonical representation, parabolic subgroup

## 1 Introduction

A Coxeter group $(W, S)$ is a group with a presentation of the form,

$$W = \mathrm{gp}\langle \{s | s \in S\} | (st)^{m_{st}} = 1 \text{ for all } s, t \in S\rangle, \tag{1.1}$$

where $m_{st} = m_{ts}$ is a positive integer or $\infty$, and $m_{st} = 1$ if and only if $s = t$ (A "relation" $(st)^\infty = 1$ is interpreted as vacuous). The cardinality $|S|$ of $S$ is called the rank of $W$. The length $l(w)$ of an element $w \in W$ is the smallest number $m$ where $w$ can be expressed as a product of $m$ elements (counting repetitions) in $S$. We are mainly interested in Coxeter groups of finite rank and assume $|S|$ is finite in this note, although some statements are still valid for infinite rank situation.

Given a Coxeter group $W$ defined as above, for a subset $I$ of $S$, let $W_I$ be the subgroup generated by $s \in I$ and call it a special subgroup of $W$. At the extremes, $W_\emptyset = \{1\}$ and $W_S = W$. For any $w \in W$, $wW_Iw^{-1}$ is called a parabolic subgroup of $W$. The parabolic closure $\mathrm{Pc}(A)$ of a subset $A$ of $W$ is defined to be the intersection of all parabolic subgroups containing $A$. It is believed that a parabolic closure is a parabolic subgroup, for example, by studying the parabolic closure of some particular element or a subgroup of a Coxeter group, D. Krammer [4] obtained some very interesting results of irreducible, infinite Coxeter groups. However, I have not seen a proof in the literature that a parabolic closure which, by definition, is the intersection of a collection of parabolic subgroups, must be a parabolic subgroup. Perhaps the result closest to this aim is

**Theorem 1.1.** *The intersection of two parabolic subgroups of a Coxeter group is a parabolic subgroup.*

This result appears in geometric form in [6] and a proof using algebraic argument is given in [5]. However the above proof does not establish the conclusion that a parabolic closure is a parabolic subgroup.

In this note, first I give a short proof of this theorem using standard facts of canonical representations of Coxeter groups (see [1] [3]). The proof has a simple and clear geometric meaning. Following this, I use the general notion of root systems developed by V. Deodhar [7] to establish some technical lemmas on the root system of a parabolic subgroup and use them to prove that the parabolic closure of any subset of $W$ is a parabolic subgroup and give an alternate description of it. All these accounts are in Section 3.

# 2 Preliminaries

In this section we collect a few basic facts about the canonical representations of Coxeter groups. The materials are taken from Chapter 5 of [3]. Let $V$ be a vector space over $\mathbf{R}$, having basis $\{\alpha_s | s \in S\}$ in one-to-one correspondence with $S$. Define a symmetric bilinear form ( , ) on $V$ by setting

$$(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}. \tag{2.1}$$

The value on the right-hand side is interpreted to be $-1$ when $m_{st} = \infty$. Now for each $s \in S$, define a linear transformation $\sigma_s : V \to V$ by $\sigma_s \lambda = \lambda - 2(\alpha_s, \lambda)\alpha_s$. Then $\sigma_s$ is an affine reflection, which has order 2 and fixes the hyperplane $H_s = \{\delta \in V | (\delta, \alpha_s) = 0\}$ pointwise, and $\sigma_s \alpha_s = -\alpha_s$. We have

**Theorem 2.1.** *There is a unique homomorphism $\sigma : W \to GL(V)$ sending $s$ to $\sigma_s$. This homomorphism is a faithful representation of $W$ and the group $\sigma(W)$ preserves the bilinear form defined as above. Moreover, for each pair $s, t \in S$, the order of $st$ in $W$ is precisely $m_{st}$.*

Now we introduce the **root system** $\Phi$ of $W$, which is defined to be the collection of all vectors $w(\alpha_s)$, where $w \in W$ and $s \in S$. An important fact about the root system is that any root $\alpha \in \Phi$ can be expressed as

$$\alpha = \sum_{s \in S} c_s \alpha_s,$$

where all the coefficients $c_s \geq 0$ (we call $\alpha$ positive and write $\alpha > 0$), or all the coefficients $c_s \leq 0$ (call $\alpha$ negative and write $\alpha < 0$). Write $\Phi^+$ and $\Phi^-$ for the respective sets of positive and negative roots. Then $\Phi^+ \bigcap \Phi^- = \emptyset$ , $\Phi^+ \bigcup \Phi^- = \Phi$ and $\Phi^- = -\Phi^+$. The map from $\Phi$ to $R = \{wtw^{-1} | w \in W, t \in S\}$ (the set of reflections in $W$) given by $\alpha = w(\alpha_s) \mapsto wsw^{-1}$ is well-defined and restricts to a bijection from $\Phi^+$ ($\Phi^-$) to $R$, and $\sigma(wsw^{-1}) = t_\alpha$, where $t_\alpha \lambda = \lambda - 2(\alpha, \lambda)\alpha$. The following fact is important to us.

**Proposition 2.2.** *Let $w \in W$, $\alpha \in \Phi^+$. Then $l(wt_\alpha) > l(w)$ if and only if $w(\alpha) > 0$.*

With the representation $\sigma : W \to \mathrm{GL}(V)$ in mind, we define a dual representation $\sigma^* : W \to \mathrm{GL}(V^*)$ as follows (and we abuse the notations by identifying $w$ with $\sigma(w)$ or $\sigma^*(w)$),

$$\langle w(f), \lambda \rangle = \langle f, w^{-1}(\lambda) \rangle \text{ for } w \in W, f \in V^*, \lambda \in V,$$

where $V^*$ is the dual space of $V$ and the natural pairing of $V^*$ with $V$ is denoted by $\langle f, \lambda \rangle$. This dual representation induces an action of $W$ on the Tits cone defined as follows. For $I \subset S$, write

$$C_I = \{f \in V^* | \langle f, \alpha_s \rangle > 0 \text{ for } s \in S - I \text{ and } \langle f, \alpha_s \rangle = 0 \text{ for } s \in I\}.$$

Notice that $C_S = \{0\}$ and write $C = C_\emptyset$, $\overline{C} = \bigcup_{I \subset S} C_I$. Define $U$ to be the union of all $w(\overline{C})$, $w \in W$. $U$ is a cone in $V^*$, called the Tits cone of $W$.

**Theorem 2.3.** (a) *Let $w \in W$ and $I, J \subset S$. If $w(C_I) \bigcap C_J \neq \emptyset$, then $I = J$ and $w \in W_I$, so $w(C_I) = C_I$. In particular, $W_I$ is the precise stabilizer in $W$ of each point of $C_I$, and $w(C_I)$, where $w \in W$, $I \subset S$, form a partition of the Tits cone $U$.*

(b) *$\overline{C}$ is a fundamental domain for the action of $W$ on $U$: the $W$-orbit of each point of $U$ meets $\overline{C}$ in exactly one point.*

(c) *The Tits cone $U$ is convex, and every closed line segment in $U$ meets just finitely many of the sets of the family $\{w(C_I) | I \subset S\}$.*

Both $\sigma$ and $\sigma^*$ are called canonical representations.

# 3 Root system of a parabolic subgroup and the parabolic closure of a set

Let me show that Theorem 2.3 implies Theorem 1.1 immediately.

*Proof of Theorem 1.1.* Given two parabolic subgroups $G_1$ and $G_2$ of $W$. Pick $x_i \in U$, $i = 1, 2$, such that $G_i$ is the stabilizer of $x_i$. Then $G_1 \bigcap G_2$ fixes the line segment $\overline{x_1 x_2}$. By (c) of Theorem 2.3, there exist $y_1 \neq y_2$ on $\overline{x_1 x_2}$ such that they belong to the same $w(C_I)$. So $y_1$ and $y_2$ have the same stabilizer $P = wW_I w^{-1}$. Now $P$ fixes the line segment $\overline{x_1 x_2}$ and hence $P \subset G_i$, $P \subset G_1 \bigcap G_2$. Since $G_1 \bigcap G_2$ fixes $\overline{x_1 x_2}$, the reversed inclusion is obvious. This completes the proof.

Now we describe a lemma on the root system $\Phi_I$ of a special subgroup $W_I$, where $\Phi_I = \{w(\alpha_s) | w \in W_I, s \in I\}$.

**Lemma 3.1.** $\Phi_I = \Phi \bigcap \mathrm{span}\{\alpha_s | s \in I\}$. *Here* span *means* $\mathbf{R}$*-span.*

It is obvious that $\Phi_I \subset \Phi \bigcap \mathrm{span}\{\alpha_s | s \in I\}$. When $W$ is finite, arguments similar to that given on page 11 of [3] yields the reversed inclusion. In the case that $W$ is of finite rank, the nontrivial fact that $\sigma^*(W)$ is a discrete subgroup of $\mathrm{GL}(V^*)$ implies that $\Phi$ is a discrete set of $V$, which makes similar arguments work. However, Lemma 3.1 holds even when $|S| = \infty$, as the following proof demonstrates. In fact, it follows from the basic properties of Coxeter groups.

*Proof of Lemma 3.1.* We want to prove that $\Phi \bigcap \mathrm{span}\{\alpha_s | s \in I\} \subset \Phi_I$. Pick an arbitray $\phi \in \Phi \bigcap \mathrm{span}\{\alpha_s | s \in I\}$, $\phi > 0$. Write $\phi = c_1 \alpha_{s_1} + \cdots + c_n \alpha_{s_n}$,

where $c_i > 0$, $s_i \in I$, $i = 1, \cdots, n$, $s_i \neq s_j$ when $i \neq j$. We assume $n \geq 2$, otherwise $\phi = \alpha_{s_1} \in \Phi_I$. Now use induction on the length $l(t_\phi)$ of $t_\phi$ . Recall from Section 2 that $t_\phi(\lambda) = \lambda - 2(\phi, \lambda)\phi$.

Notice that $1 = (\phi, \phi) = \sum_{j=1}^{n} c_j(\phi, \alpha_{s_j})$, we know $(\phi, \alpha_{s_i}) > 0$ for some $i$. A simple calculation shows that $s_i t_\phi s_i = t_{s_i(\phi)}$ and we want to show $l(s_i t_\phi s_i) < l(t_\phi)$. First it follows from

$$t_\phi(\alpha_{s_i}) = \alpha_{s_i} - 2(\phi, \alpha_{s_i})\phi < 0 \tag{3.1}$$

(we assume $n \geq 2$) that $l(t_\phi s_i) = l(t_\phi) - 1$ by Proposition 2.2 and hence $l(s_i t_\phi) = l(t_\phi) - 1$. If $s_i t_\phi(\alpha_{s_i}) > 0$, then (3.1) implies that $t_\phi(\alpha_{s_i}) = -\alpha_{s_i}$, i.e., $\alpha_{s_i} - 2(\phi, \alpha_{s_i})\phi = -\alpha_{s_i}$, hence $\phi = \alpha_{s_i}$, a contradiction to the assumption $n \geq 2$. Therefore $s_i t_\phi(\alpha_{s_i}) < 0$ and $l(t_{s_i(\phi)}) = l(s_i t_\phi s_i) = l(s_i t_\phi) - 1 = l(t_\phi) - 2$ and induction hypothesis now applies and the proof is completed.

Now we start to discuss parabolic closures. With the notations of Section 2, write $\Delta_K = \{\alpha_s | s \in K\}$ for $K \subset S$.

**Lemma 3.2.** *If $W_I = wW_Jw^{-1}$ for some $w \in W$, $I, J \subset S$, then $|I| = |J|$, and $w_0(\Delta_J) = \Delta_I$ for some $w_0 \in wW_J$, so $I = w_0 J w_0^{-1}$.*

This lemma is stated and proved in Section 3.4 of [2]. The proof given there is mainly combinatorial (without using root system), although some topological considerations (of connected components separated by some "walls" of the corresponding Cayley graph) are used. Here we give another proof.

*Proof of Lemma 3.2.* We employ a few basic facts of Coxeter groups. First, if $xtx^{-1} \in W_K$, where $x \in W$, $t \in S$ and $K \subset S$, then $xtx^{-1} = w_1 s w_1^{-1}$ for some $w_1 \in W_K$ and $s \in K$; that is, if a reflection of a Coxeter group $W$ lies in a special subgroup $W_K$, it is indeed a reflection in $W_K$ (considering $W_K$ as a Coxeter group by itself). Second, $wW_J = w_0 W_J$, where $w_0$ satisfies that $l(w_0 t) = l(w_0) + 1$ for any $t \in J$, i.e., $w_0$ is the shortest element in $wW_J$.

Now using the above $w_0$, we have $W_I = w_0 W_J w_0^{-1}$. It follows from the correspondence of root system and reflections of Coxeter group $W$ that $\Phi_I = w_0(\Phi_J)$. Comparing the maximal numbers of linearly independent positive roots in these sets (By the choice of $w_0$, $w_0(\alpha_t) > 0$, for $t \in J$.), we have $|I| = |J|$. The fact $\Phi_I = w_0(\Phi_J)$ implies each element of $\Phi_I$ is a positive or negative linear combination of $w_0(\Delta_J)$, so $\Delta_I = w_0(\Delta_J)$ and the conclusion of lemma follows.

**Lemma 3.3.** *If $W_I \not\subseteq wW_Jw^{-1}$, $wW_Jw^{-1} \not\subseteq W_I$, then $W_I \bigcap wW_Jw^{-1} = xW_Kx^{-1}$ with $|K| < \min\{|I|, |J|\}$.*

*Proof of Lemma 3.3.* The statement that $W_I \bigcap wW_Jw^{-1} = xW_Kx^{-1}$ for some $x \in W$ and $K \subset S$ is guaranteed by Theorem 1.1. Since $xW_Kx^{-1} = x_0W_Kx_0^{-1} \subset W_I$, where $x_0$ is the shortest element in $xW_K$, any root corresponding to a reflection in $x_0W_Kx_0^{-1}$ lies in $\Phi_I$, that is, $x_0(\Phi_K) \subset \Phi_I$. Comparing the maximal numbers of linearly independent positive roots in these sets, we have $|K| \leq |I|$.

Notice that $x_0(\Phi) = \Phi$, it follows from Lemma 3.1 that

$$x_0(\Phi_K) = x_0(\Phi \bigcap \text{span}\Delta_K) = \Phi \bigcap \text{span}\{x_0(\Delta_K)\}.$$

If $|K| = |I|$, noticing that $x_0(\Delta_K) \subset \Phi_I$ and $\text{span}\{x_0(\Delta_K)\} \subset \text{span}\Delta_I$, we would have

$$\Phi_I = \Phi \bigcap \text{span}\Delta_I = \Phi \bigcap \text{span}\{x_0(\Delta_K)\} = x_0(\Phi_K),$$

and hence $\Delta_I = x_0(\Delta_K)$, $W_I = x_0W_Kx_0^{-1} = xW_Kx^{-1}$, contradicting the assumption of the lemma. Hence $|K| < |I|$. Similarly, $|K| < |J|$.

Now another description of parabolic closure is

**Theorem.** *The parabolic closure* $\mathrm{Pc}$*(A) of a subset A of W is the parabolic subgroup $wW_Jw^{-1}$ containing A, with $|J|$ being the smallest.*

The proof is obvious. The statement that the above mentioned parabolic subgroup is contained in any parabolic subgroup containing $A$ follows from Lemma 3.3 and the fact (whose proof is essentially contained in the proof of Lemma 3.3): if $xW_Kx^{-1} \subset W_I$, then $|K| \leq |I|$ and if $xW_Kx^{-1}$ is a proper subgroup of $W_I$, then $|K| < |I|$.

# References


[1] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 4-6*, Springer-Verlag, New York, 2002

[2] M. Davis, *Geometry and Topology of Coxeter Groups*, in preparation.

[3] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990

[4] D. Krammer, *The conjugacy problem for Coxeter groups*, Ph. D. Thesis, Universiteit Utrecht, 1994.

[5] L. Solomon, A Mackey formula in the group ring of a Coxeter group, *J. algebra* **41** (1976), 255-268.

[6] J. Tits, *Buildings of Spherical Type and Finite* BN-*pairs*, Lecture Notes in Mathematics **386**, 1974.

[7] V.V. Deodhar, On the root system of a Coxeter group, *Comm. Algebra* **10** (1982), 611-630.